0.1 Conic sections

We choose a standard Cartesian coordinate system with coordinates (x, y).

0.1.1 The parabola

The parabola can be characterized as the locus of points equidistant from the *focus* $F(0,0)$ and the *directrix*, the line $x = -p$. Thus $x + p = \sqrt{x^2 + y^2}$ or

$$y^2 = p(2x + p).$$

0.1.2 The ellipse

Definition 1. The ellipse can be characterized as the locus of points the sum of whose distances from the two *foci* $F(-c, 0)$ and $F'(c, 0)$ is $2a$, where $a > c > 0$. Thus

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad b^2 = a^2 - c^2.$$

Here a is called the *semi-major axis*, $b > 0$ is the *semi-minor axis* and $e = \frac{c}{a}$ is the *eccentricity*. Note that $0 < e < 1$.

Definition 2. The ellipse can be characterized as the locus of points whose distance from the focus $F(0,0)$ is e times the distance from the directrix $x = -p$. Thus $e(x + p) = \sqrt{x^2 + y^2}$ or

$$\frac{(x')^2}{a^2} + \frac{(y')^2}{b^2} = 1,$$

where

$$x' = x - \frac{pe^2}{1 - e^2}, \quad y' = y,$$

and

$$a = \frac{ep}{1 - e^2}, \quad b = \frac{ep}{\sqrt{1 - e^2}}, \quad c = \frac{e^2 p}{1 - e^2}.$$

Note that this is just a translated version of the result in Definition 1, where the focus $F(-c, 0)$ in the x', y' coordinates has been moved to $F(0,0)$ in the x, y coordinates.

0.1.3 The hyperbola

Definition 1. The hyperbola can be characterized as the locus of points the difference of whose distances from the two *foci* $F(-c,0)$ and $F'(c,0)$ is $\pm 2a$, where $c > a > 0$. Thus $\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} = \pm 2a$ or

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad b^2 = c^2 - c^2.$$

Here $e = \frac{c}{a}$ is the *eccentricity*. Note that $1 < e$.

Definition 2. The right-hand branch of the hyperbola can be characterized as the locus of points whose distance from the focus $F(0,0)$ is e times the distance from the directrix $x = -p$. Thus $e(x+p) = \sqrt{x^2+y^2}$ or

$$\frac{(x')^2}{a^2} - \frac{(y')^2}{b^2} = 1,$$

where

$$x' = x - \frac{pe^2}{1-e^2}, \quad y' = y,$$

and

$$a = \frac{ep}{e^2-1}, \quad b = \frac{ep}{\sqrt{e^2-1}}, \quad c = \frac{e^2 p}{e^2-1}.$$

Note that this is a translated version of the result in Definition 1, where the focus $F(-c,0)$ in the x', y' coordinates has been moved to $F(0,0)$ in the x, y coordinates.

0.2 Conic sections in polar coordinates

We use Definition 2 for the ellipse and the hyperbola and our standard definition of the parabola. Thus a focus is at the origin, $F(0,0)$, and in each case the equation of the conic section is

$$e(x+p) = \sqrt{x^2+y^2}.$$

Now change to polar coordinates $[r, \theta]$:

$$x = r\cos\theta, \quad y = r\sin\theta,$$

so that

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}.$$

Then the equation for the conic sections is

$$e(r\cos\theta + p) = r,$$

or

$$r = \frac{ep}{1 - e\cos\theta}.$$

Special cases:

1. $e \to 0$ and $ep \to r_0$. This is the circle of radius r_0: $r = r_0$.

2. $0 < e < 1$. The ellipse. The closest approach of the ellipse to the focus, *perihelion* is $r = ep/(1+e)$ and occurs for $\cos\theta = -1$. The greatest distance from the focus, *aphelion* is $r = ep/(1-e)$ and occurs for $\cos\theta = -1$.

3. $e = 1$. The parabola. As $\cos\theta \to 1$ the point on the parabola recedes arbitrarily far from the focus. The closes approach is a distance $p/2$ and occurs for $\cos\theta = -1$.

4. $1 < e$. The hyperbola. The closest approach of the hyperbola to the focus is $r = ep/(1+e)$ and occurs for $\cos\theta = -1$. As $\cos\theta \to 1/e$ from below, the point on the hyperbola recedes arbitrarily far from the focus.

0.3 Kepler's laws of planetary motion

Kepler's 1st law: The orbit of each planet is an ellipse with the Sun at one focus.

Kepler's 2nd law: The line segment joining a planet to the Sun sweeps out equal areas in equal times.

Kepler's 3rd law: The square of the period of revolution of a planet is proportional to the cube of the semi-major axis of the planet's elliptical orbit. (The proportionality constant is independent of the planetary size and the eccentricity of the orbit.)

0.4 How Newton could derive the gravitational force from Kepler's laws

The gravitational force $\mathbf{F}$ exerted on a body of mass m by a body of mass M is given by

$$\mathbf{F} = \frac{GmM}{r^2}\hat{\mathbf{r}} = \frac{GmM}{r^3}\mathbf{r}$$

where $\mathbf{r}$ is the vector with initial point at the center of the mass m body and terminal point at the center of the mass M body, and

$$r = ||\mathbf{r}||, \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r}, \quad ||\hat{\mathbf{r}}|| = 1.$$

Here G is the gravitational constant, a universal constant that must be measured, i.e., there is no accepted theory that enables one to compute G from more basic principles. Most calculus treatments of the gravitational force, including the one in our text, show how Kepler's laws can be derived from the gravitational force. This, however, doesn't show that it is impossible to obtain Kepler's laws from some other force directed between the bodies. Here we demonstrate that Kepler's laws are exactly what is needed to derive Newton's formula for the gravitational force and a way, using modern methods and notation, that he could have discovered the force.

Kepler's 1st law. From our treatment of conic sections we see immediately that Kepler's first law implies that, in the plane of the orbit of a planet, the trajectory of the planet is given by the equation in polar coordinates

$$r(\theta(t)) = \frac{ep}{1 - e\cos\theta(t)} \tag{1}$$

where the origin is located at the center of the Sun. Here the eccentricity e and the directrix location p are to be determined, though we know that $0 \leq e < 1$. This gives us the path of the planet, but to know the exact trajectory we would need to determine the angle θ as a function of time t.

Kepler's 2nd law. The area $A(t)$ swept out by the planet between times 0 and t is given by the formula

$$A(t) = \frac{1}{2}\int_{\theta(0)}^{\theta(t)} r^2(\theta)\ d\theta$$

where $r(\theta)$ is given by (1). Kepler's 2nd law says that

$$\frac{dA(t)}{dt} = c,$$

a constant. Now, by the Fundamental Theorem of Calculus and the chain rule,

$$\frac{dA(t)}{dt} = \frac{1}{2} r^2(\theta)\theta'(t) = c, \tag{2}$$

independent of time. (As we will see later in the course, this formula can be interpreted as conservation of angular momentum.)

Let's compute c in terms of more familiar quantities. As the planet goes through one complete rotation of the Sun the angle θ increases by 2π radians and the time increases by the *period* T. Thus $\theta(T) = \theta(0) + 2\pi$. The area swept out by the planet in one full period is just the area of the ellipse:

$$A(T) = \pi ab$$

where a, b are the lengths of the semi-major and semi-minor axes, respectively. Clearly,

$$c = \frac{A(T)}{T},$$

so, we have the differential equation

$$\theta'(t) = \frac{2A(T)}{Tr^2(\theta)}. \tag{3}$$

Newton's law says that if a force $\mathbf{F}$ is exerted on a body of mass m and position $\mathbf{r}(t)$, the body moves according to the equation $\mathbf{F} = m\mathbf{r}''(t)$. From equation (1) in polar coordinates we see that the position vector for the planet is

$$\mathbf{r}(t) = x(t)\ \mathbf{i} + y(t)\ \mathbf{j} = \frac{ep}{1 - e\cos\theta}(\cos\theta\ \mathbf{i} + \sin\theta\ \mathbf{j}).$$

Thus

$$\mathbf{r}'(t) = \left[\frac{-ep\sin\theta}{(1 - e\cos\theta)^2}(\cos\theta\ \mathbf{i} + \sin\theta\ \mathbf{j}) + \frac{ep}{1 - e\cos\theta}(-\sin\theta\ \mathbf{i} + \cos\theta\ \mathbf{j})\right]\theta',$$

so, substituting (3) for θ' and simplifying, we find

$$\mathbf{r}'(t) = \frac{2A(T)}{pT}\left[-\frac{\sin\theta}{e}\,\mathbf{i} + (-1 + \frac{\cos\theta}{e})\,\mathbf{j}\right].$$

Differentiating a second time, and again substituting (3) for θ' and simplifying, we find

$$\mathbf{r}''(t) = -\frac{4A^2(T)}{epT^2r^2}(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) = -\frac{4A^2(T)}{epT^2r^2}\,\hat{\mathbf{r}}.$$

This shows that the acceleration of the planet is always toward the Sun and proportional to $1/r^2$. It does not yet give the law of gravitation because the proportionality constant $4A^2(T)/epT^2$ appears to depend on details of the orbit, such as the eccentricity and the period.

Kepler's 3rd law. In our notation, Kepler's 3rd law says that

$$T = \alpha a^{\frac{3}{2}}$$

where α is independent of the eccentricity of the orbit and the size of the planet. Recall that the semi-major and semi-minor axes of an ellipse are related to the eccentricity and directrix via

$$a = \frac{ep}{1-e^2}, \quad b = \frac{ep}{\sqrt{1-e^2}}.$$

We can solve for p from the first of these equations to get

$$p = \frac{a(1-e^2)}{e},$$

so

$$b = a\sqrt{1-e^2}.$$

Writing $A(T)$ is terms of a and e we have

$$A(T) = \pi ab = \pi a^2\sqrt{1-e^2}.$$

Substituting these results into the proportionality constant for the gravitation equation to express all orbit quantities in terms of a and e alone, we obtain

$$\frac{4A^2(T)}{epT^2} = \frac{4\pi^2a^4(1-e^2)}{a(1-e^2)\alpha^2a^3} = \frac{4\pi^2}{\alpha^2} = k,$$

a constant independent of the orbital parameters. Thus

$$\mathbf{r}''(t) = -\frac{k}{r^2}\hat{\mathbf{r}}.$$

Now from Newton's law of motion

$$\mathbf{F} = m\mathbf{r}''(t) = -\frac{mk}{r^2}\hat{\mathbf{r}},$$

so the gravitational force depends linearly on the mass m of the planet. From Newton's principle that to every action there is an equal and opposite reaction it follows that the planet exerts an equal force on the Sun, but in the opposite direction. Thus the proportionality constant must also depend linearly on the mass M of the sun. We conclude that $k = MG$ and the gravitational force is

$$\mathbf{F} = m\mathbf{r}''(t) = -\frac{mMG}{r^2}\hat{\mathbf{r}},$$

where G is a universal constant.

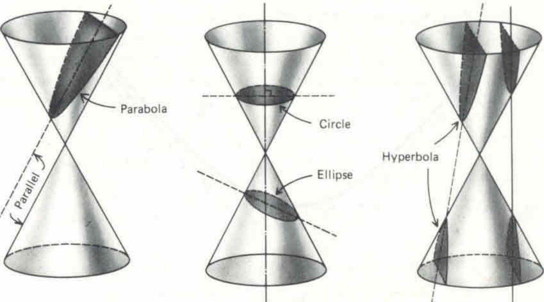
Exercises:

Problem 1 *Halley's comet has an elliptical orbit of eccentricity* 0.97 *with one focus at the Sun. It was last at perihelion in 1986 and will return in 2062. The semi-major axis of the orbit has been calculated as* 18.09 *AU where 1 AU (Astronomical Unit) is the mean distance from the Earth to the Sun, about 93 million miles. Compute the polar equation of the comet's orbit and the maximum distance from the comet to the Sun.*

Problem 2 *The speed of a planet in its orbit about the Sun is* $v = ||\mathbf{r}'(t)||$. *Show that*

$$v^2 = k(\frac{2}{r} - \frac{1}{a})$$

where a is the length of the semi-major axis. (In Newtonian mechanics, this formula is an expression of conservation of energy.)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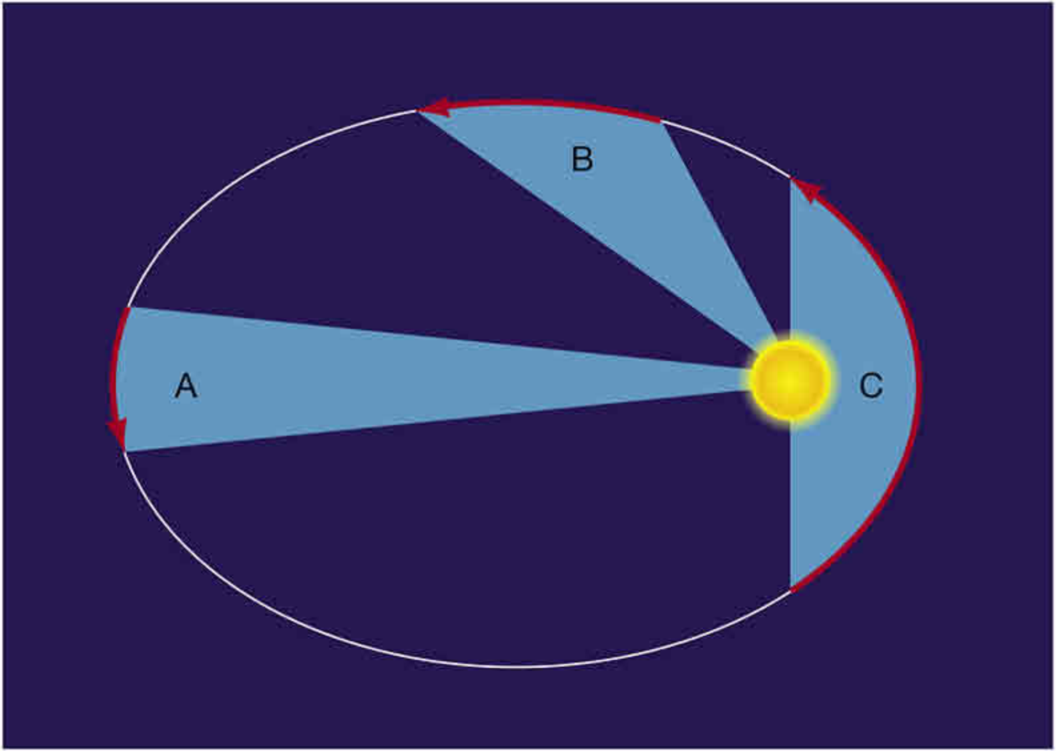
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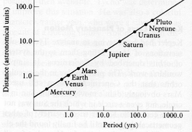

100.0
10.0
1.0
Distance (astronomical units)
Pluto
Neptune
Uranus
Saturn
Jupiter
Mars
Earth
Venus
Mercury
1.0
10.0
100.0
1000.0
Period (yrs)