GÖDEL INCOMPLETENESS

Fix an alphabet containing

- uppercase Roman: A, B, C, . . . , Z
- digits: 0,1,2,3,4,5,6,7,8,9
- some special characters: ⟨blank⟩ and ⟨return/line feed⟩ and ⟨comma⟩
- other characters needed for Pascal, *e.g.*, $(\ ) < > \ : \ ; + - * / =$
- a "delimiter" character: $\dashv$

and **not** containing

- lowercase Roman: a, b, c, . . . , z
- lowercase italic: $a, b, c, \ldots, z$
- lowercase Greek: $\alpha, \beta, \gamma, \ldots, \omega$

$\{\surd\}$ A **string** is a sequence of characters from the alphabet which

- is finite
- does not contain $\dashv$.

$m = 2$
$\sigma =$TIMES
$m\ \sigma\ m$ IS FOUR=2 TIMES 2 IS FOUR

Strings are never true or false.
However, they may be "theorems".

Some strings are **integers**.

- 2 is.
- FOUR is not.
- 35 and -27 are.
- XXXV is not.

Extend Pascal: Include a command

WRITE_DELIMITER;

which outputs: $\dashv$

$\{\surd\}$ Write a Pascal program, our "**string generator**", which outputs:

$$\sigma_1 \dashv \sigma_2 \dashv \sigma_3 \dashv \cdots$$

such that every possible string appears at least once among the strings

$$\sigma_1, \sigma_2, \ldots.$$

The **number** of a string σ is

$$\min\{n \,|\, \sigma = \sigma_n\}.$$

$\{\surd\}$ If u is the number of a string σ, then $u \in \mathbb{Z}$, $u > 0$ and $\sigma_u = \sigma$.

$\{\surd\}$ A string σ is **organized** if there are strings ρ and μ such that μ has no parentheses and such that

$$\sigma = (\rho)(\mu)$$

In this case ρ and μ are called, respectively, the **preface** and the **main body** of σ. They are determined by σ.

$\{\surd\}$ Example:

((()X)(7VQWZ) is organized.

The main body is: 7VQWZ
The preface is: (()X

Substitution notation: If

$$\sigma = \mathrm{W)!W}$$

and

$$n = 35$$

then

$$\sigma|\mathrm{W} \rightarrow n = 35)!35$$

and is obtained by replacing every occurrence of W in σ with the string 35.

A **P-function** is the code for a Pascal function, written all in caps, such that:

- input is a single integer;
- output is a single integer; and
- no compile errors, no runtime errors, no infinite loops.

Some strings are P-functions.

Let α be the 55-character string:

```
FUNCTION
  OP(IP:INTEGER):INTEGER;
BEGIN
OP:=IP*IP;
END;
```

Then α is a P-function.

For any P-function σ, for $n \in \mathbb{Z}$, let

$$\sigma[\![n]\!]$$

be the output from inputting n into the function σ; then $\sigma[\![n]\!] \in \mathbb{Z}$.

For example, with α defined above,

$$\alpha[\![2]\!] = 4.$$

Formal Theories

$\{\surd\}$ A **formal theory** consists of two Pascal programs: A **math string tester** and a **theorem generator**.

Some strings are **math**, as determined by a math string tester, which has a string as input, and has only two possible outputs:

"This is math" or "This is not math".

$\{\surd\}$ A theorem generator has output

$$\pi_1 \dashv \pi_2 \dashv \pi_3 \dashv \cdots$$

where $\pi_1, \pi_2, \pi_3, \ldots$ are math strings called **theorems**.

Some math strings are theorems.

One typical assumption:

$\{\surd\}$ **I. Complete and consistent.** Assume, for any math string σ, that

either σ or NOT(σ)

is a theorem, but *not* both.

By itself this is easily attained, if we forego the need that, in some form or another, statements like "$2 \times 2 = 4$" appear in the list of theorems produced by the theorem generator.

So we typically would impose another condition:

A vague requirement: Assume that the formal theory "contains" arithmetic.

A little more precise: Assume that

- any basic assertion about arithmetic can be formulated as a string; that
- the resulting string is a math string; and that
- that math string is a theorem iff the original basic assertion is true.

A "**basic assertion**" is an assertion of the form: $\alpha[\![2]\!] = 4$

Convention:
The corresponding "**basic assertion string**" is: $(\alpha)(2{-}{>}4)$

$\{\surd\}$ **II. Contains arithmetic.**
Assume, for any P-function κ, for any $s, t \in \mathbb{Z}$, that

1. $(\kappa)(s{-}{>}t)$ is a math string; and
2. $(\kappa)(s{-}{>}t)$ is a theorem iff $\kappa[\![s]\!] = t$.

For example, with α being the squaring P-function defined above, we have $\alpha[\![2]\!] = 4$. Then

$$(\alpha)(2{-}{>}4)$$

is a theorem. On the other hand,

$$(\alpha)(2{-}{>}0)$$

is math, but is not a theorem.

$\{\surd\}$ Assuming I and II, we wish to obtain a **contradiction**.

This will prove:

Gödel's Theorem.
No formal theory that contains arithmetic can be complete and consistent.

Write a "**theorem tester**" P-function τ such that, for any integer input u,

- $u \leq 0 \Longrightarrow$ o/p -1, stop;
- σ_u is not math $\Longrightarrow$ o/p -1, stop;
- σ_u is a theorem $\Longrightarrow$ o/p 1, stop;
- NOT(σ_u) is a theorem $\Rightarrow$ o/p 0, stop.

$\{\surd\}$ For any integer $u > 0$, we have: if σ_u is math, then $\tau[\![u]\!] \in \{0, 1\}$.

$\{\surd\}$ For any integer $u > 0$, we have:

$$\tau[\![u]\!] = 1$$

if and only if

$$\sigma_u \text{ is a theorem.}$$

Write a "**W replacer**" P-function ω such that, for any integer input s,

- $s \leq 0 \Longrightarrow$ o/p -1, stop;
- σ_s not organized $\Longrightarrow$ o/p -1, stop;
- let $\mu :=$ the main body of σ_s;
- let $\rho :=$ the preface of σ_s;
- COMMENT: $\quad \sigma_s = (\rho)(\mu)$
- let $\mu' := \mu|\mathrm{W} \to s$;
- let $\quad \sigma' := (\rho)(\mu')$;
- o/p the number of σ', stop.

$\{\surd\}$ *E.g.*: If s is the number of

(KBW$*$)(ZVWX5W)

then $\omega[\![s]\!]$ is the number of

(KBW$*$)(ZVsX5s)

Write a "**composite**" P-function κ such that, for all $s \in \mathbb{Z}$, $\kappa[\![s]\!] = \tau[\![\omega[\![s]\!]]\!]$.

Let s be the number of

$$(\kappa)(\mathrm{W}{-}{>}0)$$

Let $u := \omega[\![s]\!]$. Then u is the number of

$$(\kappa)(s{-}{>}0)$$

Then $u \in \mathbb{Z}$, $u > 0$, and σ_u is equal to

$$(\kappa)(s{-}{>}0)$$

By II, σ_u is a math string, and, also, σ_u is a theorem iff $\kappa[\![s]\!] = 0$.

As σ_u is math, $\tau[\![u]\!] \in \{0, 1\}$.

$$\tau[\![u]\!] = 1$$

if and only if

$$\sigma_u \text{ is a theorem}$$

if and only if

$$\kappa[\![s]\!] = 0$$

if and only if

$$\tau[\![\omega[\![s]\!]]\!] = 0$$

if and only if

$$\tau[\![u]\!] = 0$$

if and only if

$$\tau[\![u]\!] \neq 1.$$

Contradiction.